Massachusetts Tests for Educator Licensure®

TEST INFORMATION BOOKLET

47 Middle School Mathematics

MA-SG-FLD047-05

Massachusetts Department of Education

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Test Overview Chart: Middle School Mathematics (47)

Subareas		Approximate Number of Multiple-Choice Items	Number of Open-Response Items
I.	Number Sense and Operations	18–20	
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IV.	Data Analysis, Statistics, and Probability	14–16	
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The Middle School Mathematics test is designed to assess the candidate's knowledge of the subject matter required for the Massachusetts Mathematics (Level: 5–8) license. This subject matter knowledge is delineated in the Massachusetts Department of Education's *Regulations for Educator Licensure and Preparation Program Approval* (7/2001), 603 CMR 7.06 "Subject Matter Knowledge Requirements for Teachers."

The Middle School Mathematics test assesses the candidate's proficiency and depth of understanding of the subject at the level required for a baccalaureate minor (minimum of 24 semester hours). Candidates are typically nearing completion of or have completed their undergraduate work when they take the test.

The multiple-choice items on the test cover the subareas as indicated in the chart above. The open-response items may relate to topics covered in any of the subareas and will typically require breadth of understanding of the mathematics field and the ability to relate concepts from different aspects of the field. Responses to the open-response items are expected to be appropriate and accurate in the application of subject matter knowledge, to provide high-quality and relevant supporting evidence, and to demonstrate a soundness of argument and understanding of the mathematics field.

Massachusetts Tests for Educator Licensure™

FIELD 47: MIDDLE SCHOOL MATHEMATICS TEST OBJECTIVES

Subarea

	Multiple-Choice	**Range of Objectives**	**Approximate Test Weighting**
I.	Number Sense and Operations	01–04	15%
II.	Patterns, Relations, and Algebra	05–10	25%
III.	Geometry and Measurement	11–15	18%
IV.	Data Analysis, Statistics, and Probability	16–17	12%
V.	Trigonometry, Calculus, and Discrete Mathematics	18–20	10%
			80%
	Open-Response		
VI.	Integration of Knowledge and Understanding	21	**20%**

SUBAREAS:

NUMBER SENSE AND OPERATIONS
PATTERNS, RELATIONS, AND ALGEBRA
GEOMETRY AND MEASUREMENT
DATA ANALYSIS, STATISTICS, AND PROBABILITY
TRIGONOMETRY, CALCULUS, AND DISCRETE MATHEMATICS
INTEGRATION OF KNOWLEDGE AND UNDERSTANDING

NUMBER SENSE AND OPERATIONS [15%]

0001 Understand the structure of numeration systems and multiple representations of numbers.

For example: place value; number bases (e.g., base 2, base 10); order relations; relationships between operations (e.g., multiplication as repeated additions); number factors and divisibility; prime and composite numbers; prime factorization; multiple representations of numbers (e.g., physical models, diagrams, numerals); and properties of early numeration systems (e.g., Mayan, Mesopotamian, Egyptian).

0002 Understand principles and operations related to integers, fractions, decimals, percents, ratios, and proportions.

For example: order of operations; identity and inverse elements; associative, commutative, and distributive properties; absolute value; operations with signed numbers; multiple representations (e.g., area models for multiplication) of number operations; analyzing standard algorithms for addition, subtraction, multiplication, and division of integers and rational numbers; number operations and their inverses; and the origins and development of standard computational algorithms.

0003 Understand and solve problems involving integers, fractions, decimals, percents, ratios, and proportions.

For example: solving a variety of problems involving integers, fractions, decimals, percents (including percent increase and decrease), ratios, proportions, and average rate of change; and using estimation to judge the reasonableness of solutions to problems.

0004 Understand the properties of real numbers and the real number system.

For example: rational and irrational numbers; properties (e.g., closure, distributive, associative) of the real number system and its subsets; operations and their inverses; the real number line; roots and powers; the laws of exponents; scientific notation; using number properties to prove theorems (e.g., the product of two even numbers is even); and problems involving real numbers and their operations.

PATTERNS, RELATIONS, AND ALGEBRA [25%]

0005 Understand and use patterns to model and solve problems.

For example: making conjectures about patterns presented in numeric, geometric, or tabular form; representing patterns and relations using symbolic notation; identifying patterns of change created by functions (e.g., linear, quadratic, exponential); and using finite and infinite series and sequences (e.g., Fibonacci, arithmetic, geometric) to model and solve problems.

0006 Understand how to manipulate and simplify algebraic expressions and translate problems into algebraic notation.

For example: the nature of a variable; evaluating algebraic expressions for a given value of a variable; the relationship between standard computational algorithms and algebraic processes; expressing direct and inverse relationships algebraically; expressing one variable in terms of another; manipulating and simplifying algebraic expressions; solving equations; and using algebraic expressions to model situations.

0007 Understand properties of functions and relations.

For example: the difference between functions and relations; the generation and interpretation of graphs that model real-world situations; multiple ways of representing functions (e.g., tabular, graphic, verbal, symbolic); properties of functions and relations (e.g., domain, range, continuity); piecewise-defined functions; addition, subtraction, and composition of functions; and graphs of functions and their transformations [e.g., the relationships among $f(x)$, $f(x + k)$, and $f(x) + k$].

0008 Understand properties and applications of linear relations and functions.

For example: the relationship between linear models and rate of change; direct variation; graphs of linear equations; slope and intercepts of lines; finding an equation for a line; methods of solving systems of linear equations and inequalities (e.g., graphing, substitution); and modeling and solving problems using linear functions and systems.

0009 Understand properties and applications of quadratic relations and functions.

For example: methods of solving quadratic equations and inequalities (e.g., factoring, completing the square, quadratic formula, graphing); real and complex roots of quadratic equations; graphs of quadratic functions; quadratic maximum and minimum problems; and modeling and solving problems using quadratic relations, functions, and systems.

0010 Understand properties and applications of exponential, polynomial, rational, and absolute value functions and relations.

For example: problems involving exponential growth (e.g., population growth, compound interest) and decay (e.g., half-life); inverse variation; modeling problems using rational functions; properties and graphs of polynomial, rational, and absolute value functions; and the use of graphing calculators and computers to find numerical solutions to problems involving exponential, polynomial, rational, and absolute value functions.

GEOMETRY AND MEASUREMENT [18%]

0011 Understand principles, concepts, and procedures related to measurement.

For example: using appropriate units of measurement; unit conversions within and among measurement systems; problems involving length, area, volume, mass, capacity, density, time, temperature, angles, and rates of change; problems involving similar plane figures and indirect measurement; the effect of changing linear dimensions on measures of length, area, or volume; and the effects of measurement error and rounding on computed quantities (e.g., area, density, speed).

0012 Understand the principles of Euclidean geometry and use them to prove theorems.

For example: the nature of axiomatic systems; undefined terms and postulates of Euclidean geometry; relationships among points, lines, angles, and planes; methods for proving triangles congruent; properties of similar triangles; justifying geometric constructions; proving theorems within the axiomatic structure of Euclidean geometry; and the origins and development of geometry in different cultures (e.g., Greek, Hindu, Chinese).

0013 Apply Euclidean geometry to analyze the properties of two-dimensional figures and to solve problems.

For example: using deduction to justify properties of and relationships among triangles, quadrilaterals, and other polygons (e.g., length of sides, angle measures); identifying plane figures given characteristics of sides, angles, and diagonals; the Pythagorean theorem; special right triangle relationships; arcs, angles, and segments associated with circles; deriving and applying formulas for the area of composite shapes; and modeling and solving problems involving two-dimensional figures.

0014 Solve problems involving three-dimensional shapes.

For example: area and volume of and relationships among three-dimensional figures (e.g., prisms, pyramids, cylinders, cones); perspective drawings; cross sections (including conic sections) and nets; deriving properties of three-dimensional figures from two-dimensional shapes; and modeling and solving problems involving three-dimensional geometry.

0015 Understand the principles and properties of coordinate and transformational geometry.

For example: representing geometric figures (e.g., triangles, circles) in the coordinate plane; using concepts of distance, midpoint, slope, and parallel and perpendicular lines to classify and analyze figures (e.g., parallelograms); characteristics of dilations, translations, rotations, reflections, and glide-reflections; types of symmetry; properties of tessellations; transformations in the coordinate plane; and using coordinate and transformational geometry to prove theorems and solve problems.

DATA ANALYSIS, STATISTICS, AND PROBABILITY [12%]

0016 Understand descriptive statistics and the methods used in collecting, organizing, reporting, and analyzing data.

For example: constructing and interpreting tables, charts, and graphs (e.g., line plots, stem-and-leaf plots, box plots, scatter plots); measures of central tendency (e.g., mean, median, mode) and dispersion (e.g., range, standard deviation); frequency distributions; percentile scores; the effects of data transformations on measures of central tendency and variability; evaluating real-world situations to determine appropriate sampling techniques and methods for gathering and organizing data; making appropriate inferences, interpolations, and extrapolations from a set of data; interpreting correlation; and problems involving linear regression models.

0017 Understand the fundamental principles of probability.

For example: representing possible outcomes for a probabilistic situation; counting strategies (e.g., permutations and combinations); computing theoretical probabilities for simple and compound events; using simulations to explore real-world situations; connections between geometry and probability (e.g., probability as a ratio of two areas); and using probability models to understand real-world phenomena.

TRIGONOMETRY, CALCULUS, AND DISCRETE MATHEMATICS [10%]

0018 Understand the properties of trigonometric functions and identities.

For example: degree and radian measure; right triangle trigonometry; the law of sines and the law of cosines; graphs and properties of trigonometric functions and their inverses; amplitude, period, and phase shift; trigonometric identities; and using trigonometric functions to model real-world periodic phenomena.

0019 Understand the conceptual basis of calculus.

For example: the concept of limit; the relationship between slope and rates of change; how the derivative relates to maxima, minima, points of inflection, and concavity of curves; the relationship between integration and the area under a curve; modeling and solving basic problems using differentiation and integration; and the development of calculus.

0020 Understand the principles of discrete/finite mathematics.

For example: properties of sets; recursive patterns and relations; problems involving iteration; properties of algorithms; finite differences; linear programming; properties of matrices; and characteristics and applications of graphs and trees.

INTEGRATION OF KNOWLEDGE AND UNDERSTANDING [20%]

In addition to answering multiple-choice items, candidates will prepare written responses to questions addressing content from the preceding objectives, which are summarized in the objective and descriptive statement below.

0021 Prepare an organized, developed analysis on a topic related to one or more of the following: number sense and operations; patterns, relations, and algebra; geometry and measurement; data analysis, statistics, and probability; and trigonometry, calculus, and discrete mathematics.

For example: presenting a detailed solution to a problem involving one or more of the following: place value, number base, and the structure and operations of number systems; application of ratios and proportions in a variety of situations; properties, attributes, and representations of linear functions; modeling problems using exponential functions; the derivative as a rate of change and the integral as area under the curve; applications of plane and three-dimensional geometry; and design, analysis, presentation, and interpretation of a statistical survey.

FORMULAS

Description	Formula
Sum of the measures of the interior angles in a polygon	$S = (n - 2) \times 180$
Circumference of a circle	$C = 2\pi r$
Area of a circle	$A = \pi r^2$
Area of a triangle	$A = \frac{1}{2}bh$
Surface area of a sphere	$A = 4\pi r^2$
Lateral surface area of a right circular cone	$A = \pi r\sqrt{r^2 + h^2}$
Surface area of a cylinder	$A = 2\pi rh + 2\pi r^2$
Volume of a sphere	$V = \frac{4}{3}\pi r^3$
Volume of a right cone and a pyramid	$V = \frac{1}{3}Bh$
Volume of a cylinder	$V = \pi r^2 h$
Sum of an arithmetic series	$S_n = \frac{n}{2}[2a + (n - 1)d] = n\left(\frac{a + a_n}{2}\right)$
Sum of a geometric series	$S_n = \frac{a(1 - r^n)}{1 - r}$
Sum of an infinite geometric series	$\sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}, \; \lvert r \rvert < 1$
Distance formula	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Midpoint formula	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
Slope	$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$
Law of sines	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

FORMULAS (continued)

Description	Formula
Law of cosines	$c^2 = a^2 + b^2 - 2ab \cos C$
Arc length	$s = r\theta$
Density of an object	$D = \frac{m}{V}$
Quadratic formula	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Calculators will NOT be provided for the Middle School Mathematics test (Field 47).

1. **Use the sequence of steps below to answer the question that follows.**

$$\begin{aligned} 4.26 \times 2.2 &= \left(426 \times \frac{1}{100}\right) \times \left(22 \times \frac{1}{10}\right) \\ &= 426 \times \left(\frac{1}{100} \times 22\right) \times \frac{1}{10} \\ &= 426 \times \left(22 \times \frac{1}{100}\right) \times \frac{1}{10} \\ &= (426 \times 22) \times \left(\frac{1}{100} \times \frac{1}{10}\right) \\ &= (426 \times 22) \times \left(\frac{1}{1000}\right) \end{aligned}$$

The sequence of steps above could be used to answer which of the following questions?

A. How can a decimal be converted to a fraction reduced to lowest terms?

B. Where should the decimal point be placed in the product of two decimals?

C. Why is it necessary to invert and multiply when solving problems involving fractions?

D. How is scientific notation used to multiply decimals?

2. Which of the following situations represents $2\frac{2}{5} \div \frac{1}{4}$?

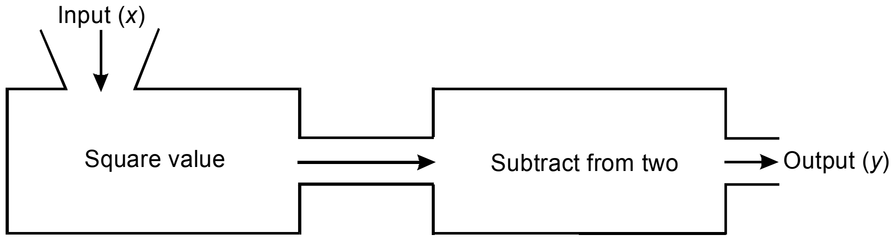
   A. A $2\frac{2}{5}$-acre lot needs to be plowed. If four workers split the plowing evenly, how many acres will each person plow?

   B. One side of a one-fourth-square-foot rectangle is $2\frac{2}{5}$ feet. How long is the other side?

   C. Alix sawed off one fourth of a $2\frac{2}{5}$-yard log. How many yards did Alix saw off?

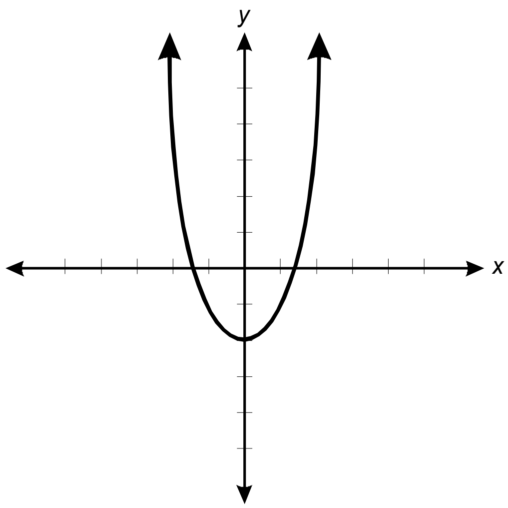
   D. Terry picked $2\frac{2}{5}$ pounds of berries and put them into one-quarter-pound containers. How many containers of berries did Terry fill?

3. **Use the diagram below to answer the question that follows.**

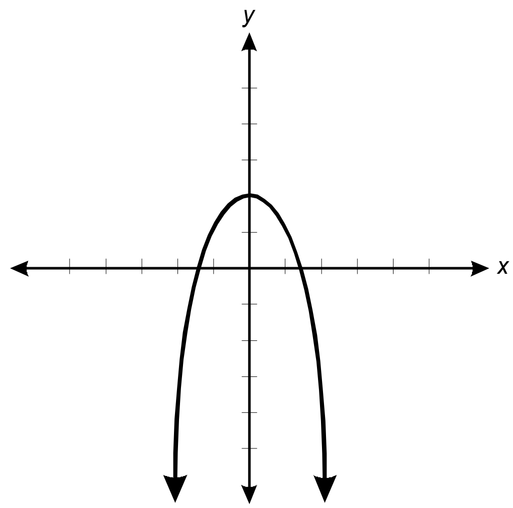


Which of the following graphs represents the function described in the above diagram?

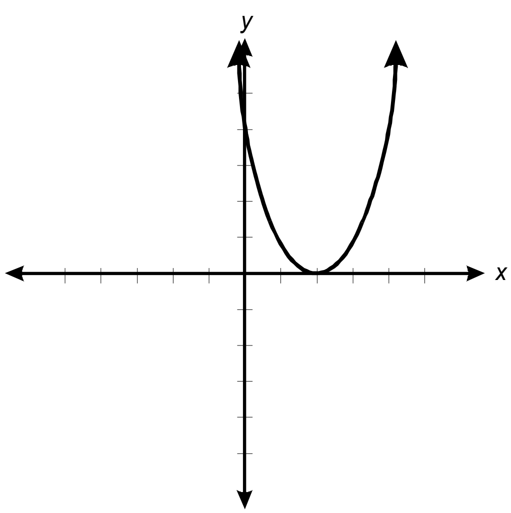
A.



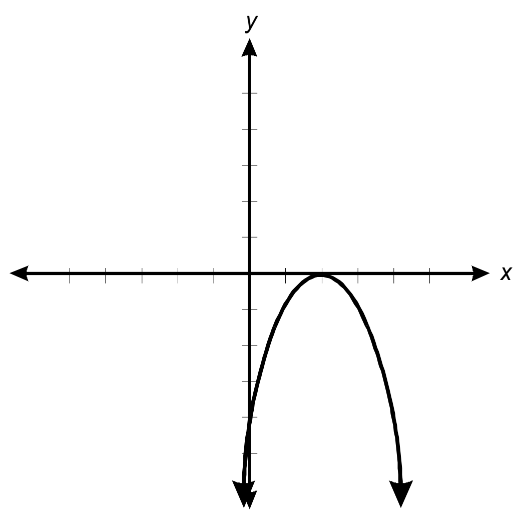
B.



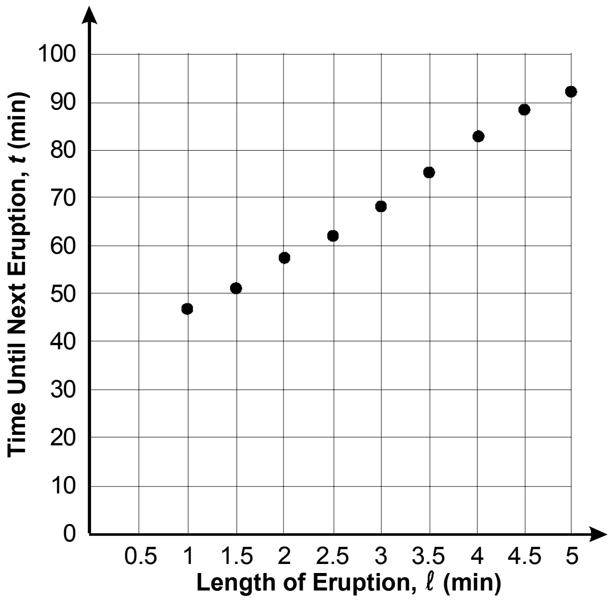
C.



D.



4. **Use the graph below to answer the question that follows.**



The graph shown above represents the relationship between the length of a geyser's eruption, ℓ, and the time until the next eruption, t. Which of the following linear equations best models the data?

A. $t + 12\ell = 33$

B. $2t - 3\ell = -66$

C. $12\ell - t = -33$

D. $3\ell + 2t = 66$

5. In solving the quadratic equation $x^2 + 14x - 4 = -30$ by completing the square, the first step is to add 4 to both sides of the equation. The second step is to:

A. add 49 to both sides of the equation.

B. factor x from the binomial $x^2 + 14x$.

C. factor the number 2 from 14, –4, and 30.

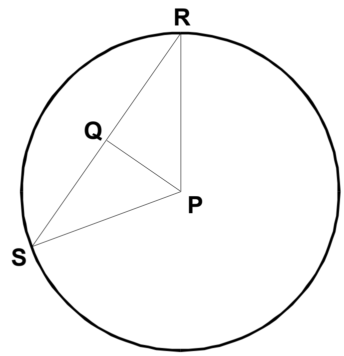
D. take the square root of both sides of the equation.

6. **Use the geometric proof below to answer the question that follows.**

Given: Circle P;

$\overline{PQ}$ is a median of $\triangle PRS$

Prove: $\triangle RPQ \cong \triangle SPQ$



Statements	**Reasons**
1. Circle P; $\overline{PQ}$ is a median of $\triangle PRS$	1. Given
2. $\overline{RQ} \cong \overline{QS}$	2. Definition of median
3. $\overline{PR}$ and $\overline{PS}$ are radii of P.	3. Definition of radius
4. $\overline{PR} \cong \overline{PS}$	4. All the radii of a circle are congruent.
5. _?_	5. _?_
6. $\triangle RPQ \cong \triangle SPQ$	6. SSS postulate

Which of the following statements and reasons would be most appropriate in step 5 of this proof?

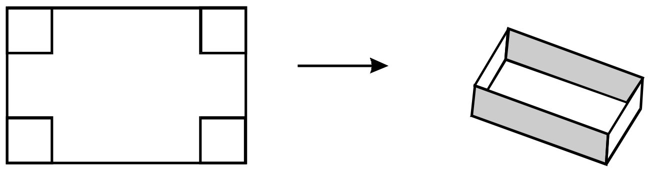
A. **Statement** $\angle QPR \cong \angle QPS$ — **Reason** $\overline{PQ} \perp \overline{RS}$

B. **Statement** P is the center of the circle. — **Reason** Definition of a circle

C. **Statement** $\angle PSQ \cong \angle PRQ$ — **Reason** Properties of isosceles triangle

D. **Statement** $\overline{PQ} \cong \overline{PQ}$ — **Reason** Reflexive property of congruence

7. **Use the diagram below to answer the question that follows.**



Square corners, 5 units on a side, are removed from a 15-unit-by-25-unit rectangular piece of cardboard. The sides of the cardboard are then folded to form an open box. Which of the following is the volume, in cubic units, of the box?

A. 375

B. 625

C. 1000

D. 1125

8. Which of the following situations best represents a random sampling?

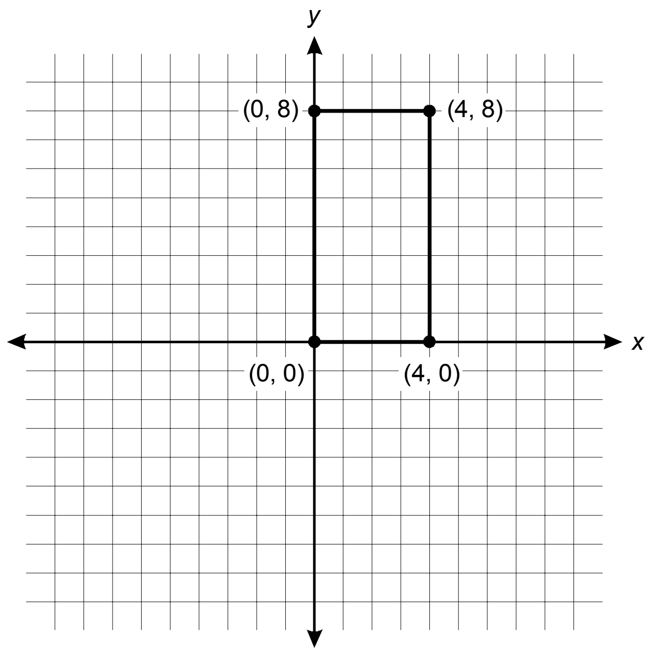
A. Ask every tenth person coming out of a health spa how many times a week they exercise to determine how often people in the town exercise.

B. Survey students in advanced biology classes to determine the average amount of time students in a certain school study each week.

C. Find the heights of all boys in a senior gym class to determine the average height of all boys in the school.

D. Count the number of chocolate chips in every fifth cookie to determine the average number of chocolate chips per cookie in a bag of cookies.

9. **Use the diagram below to answer the question that follows.**



A point is randomly selected within the rectangle shown in the diagram above. What is the probability that the *y*-coordinate of the point is less than or equal to 2?

A. $\frac{1}{8}$

B. $\frac{2}{9}$

C. $\frac{1}{4}$

D. $\frac{1}{2}$

10. The graph of the function $f(x) = 2x^2 - 6x + 4$ has its minimum value at the point $\left(\frac{3}{2}, -\frac{1}{2}\right)$. Which of the following statements must be true?

    A. The derivative of $f(x)$ is zero when x is equal to $\frac{3}{2}$.

    B. The value of $f(x)$ approaches a limit of zero as x approaches $\frac{3}{2}$.

    C. The graph of $f(x)$ shifts from a concave downward shape to a concave upward shape.

    D. The slope of the line tangent to $f(x)$ at $x = \frac{3}{2}$ is equal to $-\frac{1}{2}$.

11. **Use the information below to complete the exercise that follows.**

A company is considering two bonus-plan options for its employees for the next 20 years. The two options are explained in the following chart.

Option 1:	Receive \$2 the first year. Every year thereafter receive twice the bonus amount of the previous year.
Option 2:	Receive \$200 the first year. Every year thereafter receive \$200 more than the bonus amount of the previous year.

Use your knowledge of exponential and linear functions to develop a response in which you analyze the bonus received each year during a 20-year period under each option. In your response:

- create a data table representing the bonus received each year over a 12-year period for each option;
- graph the data from both tables on the same coordinate grid and connect the data with the line or curve that best fits the data;
- compare the bonus plans over the 12-year period, including a discussion of the significance of the point of intersection of the two graphs;
- explain what type of function, exponential or linear, models each option;
- find equations that describe each option; and
- identify an expression that represents the difference between the bonuses received under the two options in the twentieth year.

Be sure to show your work and explain the reasoning you use in analyzing and solving this problem.

Answer Key and Sample Response: Middle School Mathematics (47)

Question Number	Correct Response	Test Objective
1.	**B**	Understand principles and operations related to integers, fractions, decimals, percents, ratios, and proportions.
2.	**D**	Understand and solve problems involving integers, fractions, decimals, percents, ratios, and proportions.
3.	**B**	Understand properties of functions and relations.
4.	**C**	Understand properties and applications of linear relations and functions.
5.	**A**	Understand properties and applications of quadratic relations and functions.
6.	**D**	Understand the principles of Euclidean geometry and use them to prove theorems.
7.	**A**	Solve problems involving three-dimensional shapes.
8.	**D**	Understand descriptive statistics and the methods used in collecting, organizing, reporting, and analyzing data.
9.	**C**	Understand the fundamental principles of probability.
10.	**A**	Understand the conceptual basis of calculus.

The sample response below reflects a strong knowledge and understanding of the subject matter.

The data tables representing the bonuses received each year over a 12-year period for the two options are shown below.

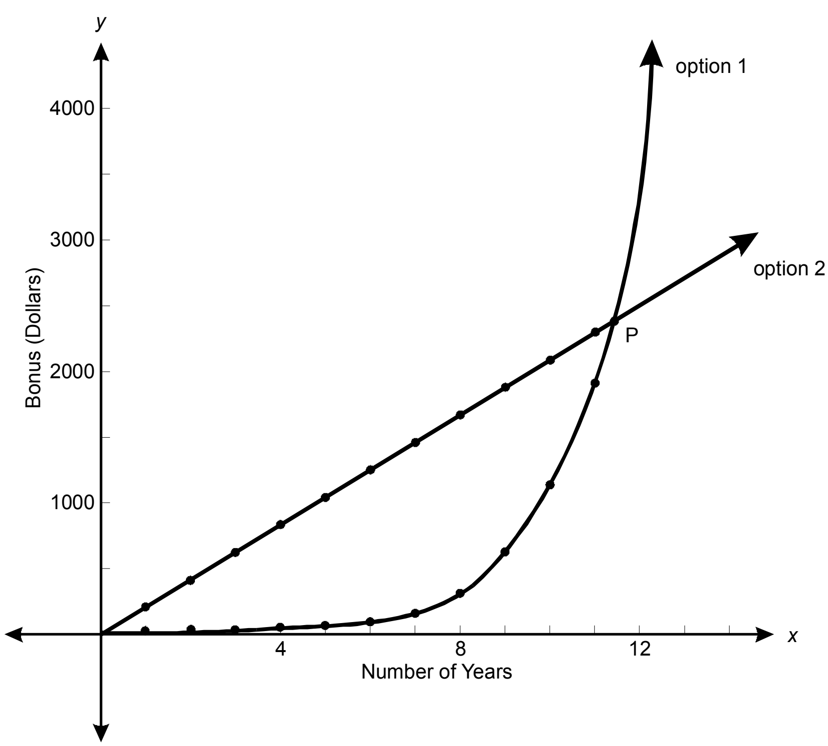
Option 1

Year	1	2	3	4	5	6	7	8	9	10	11	12
Bonus (dollars)	2	4	8	16	32	64	128	256	512	1024	2048	4096

Option 2

Year	1	2	3	4	5	6	7	8	9	10	11	12
Bonus (dollars)	200	400	600	800	1000	1200	1400	1600	1800	2000	2200	2400

The data from the tables can be graphed on a coordinate grid as shown below.



(continued)

The two graphs intersect at point P, where they both pay the same bonus. For all years before this point (approximately years 1 through 11), option 2 yields a higher bonus. For all years after this point (years 12 and beyond), option 1 yields a higher bonus.

The bonus plan offered in option 1 increases by a constant factor of 2 and therefore is modeled by an exponential equation. The exponential equation that models this bonus plan is $y = 2^x$. The bonus plan offered in option 2 increases by a constant rate of $200 each year and therefore is modeled by a linear equation. The linear equation that models this bonus plan is $y = 200x$.

In the 20th year, the bonus offered by option 1 exceeds that offered by option 2 by $2^{20} - 4000$ dollars.