Massachusetts Tests
for Educator Licensure®

TEST INFORMATION BOOKLET

09 Mathematics

MA-SG-FLD009-05

Massachusetts Department of Education

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Test Overview Chart: Mathematics (09)

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The Mathematics test is designed to assess the candidate's knowledge of the subject matter required for the Massachusetts Mathematics (Levels: 5–8; 8–12) license. This subject matter knowledge is delineated in the Massachusetts Department of Education's *Regulations for Educator Licensure and Preparation Program Approval* (7/2001), 603 CMR 7.06 "Subject Matter Knowledge Requirements for Teachers."

The Mathematics test assesses the candidate's proficiency and depth of understanding of the subject at the level required for a baccalaureate major, according to Massachusetts standards. Candidates are typically nearing completion of or have completed their undergraduate work when they take the test.

The multiple-choice items on the test cover the subareas as indicated in the chart above. The open-response items may relate to topics covered in any of the subareas and will typically require breadth of understanding of the mathematics field and the ability to relate concepts from different aspects of the field. Responses to the open-response items are expected to be appropriate and accurate in the application of subject matter knowledge, to provide high-quality and relevant supporting evidence, and to demonstrate a soundness of argument and understanding of the mathematics field.

Massachusetts Tests for Educator Licensure™

FIELD 09: MATHEMATICS
TEST OBJECTIVES

Subarea

	Multiple-Choice	**Range of Objectives**	**Approximate Test Weighting**
I.	Number Sense and Operations	01–03	12%
II.	Patterns, Relations, and Algebra	04–10	23%
III.	Geometry and Measurement	11–15	19%
IV.	Data Analysis, Statistics, and Probability	16–18	10%
V.	Trigonometry, Calculus, and Discrete Mathematics	19–23	16%
			80%
	Open-Response		
VI.	Integration of Knowledge and Understanding	24	**20%**

SUBAREAS:

NUMBER SENSE AND OPERATIONS
PATTERNS, RELATIONS, AND ALGEBRA
GEOMETRY AND MEASUREMENT
DATA ANALYSIS, STATISTICS, AND PROBABILITY
TRIGONOMETRY, CALCULUS, AND DISCRETE MATHEMATICS
INTEGRATION OF KNOWLEDGE AND UNDERSTANDING

NUMBER SENSE AND OPERATIONS [12%]

0001 Understand the structure of numeration systems and solve problems using integers, fractions, decimals, percents, ratios, and proportions.

For example: place value; order relationships; relationships between operations (e.g., division as the inverse of multiplication); multiple representations of numbers and of number operations (e.g., area models for multiplication); absolute value; signed numbers; computational algorithms; problems involving integers, fractions, decimals, percents, ratios, and proportions; the use of estimation to judge the reasonableness of solutions to problems; the origins and development of standard computational algorithms; and properties of early numeration systems (e.g., Mayan, Mesopotamian, Egyptian).

0002 Understand the properties of real and complex numbers and the real and complex number systems.

For example: rational and irrational numbers; multiple representations of complex numbers (e.g., vector, trigonometric, exponential); properties (e.g., closure, distributive, associative) of the real and complex number systems and their subsets; operations on complex numbers; the laws of exponents; calculating roots and powers of real and complex numbers; scientific notation; using number properties to prove theorems; and problems involving real and complex numbers and their operations.

0003 Understand the principles of number theory.

For example: number factors and divisibility; prime and composite numbers; prime factorization; Euclid's algorithm; congruence classes and modular arithmetic; Mersenne primes and perfect numbers; statement of Fermat's Last Theorem; and the fundamental theorem of arithmetic.

PATTERNS, RELATIONS, AND ALGEBRA [23%]

0004 Understand and use patterns to model and solve problems.

For example: conjectures about patterns presented in numeric, geometric, or tabular form; representation of patterns using symbolic notation; identification of patterns of change created by functions (e.g., linear, quadratic, exponential); iterative and recursive functional relationships (e.g., Fibonacci numbers); Pascal's triangle and the binomial theorem; and using finite and infinite sequences and series (e.g., arithmetic, geometric) to model and solve problems.

0005 Understand the properties of functions and relations.

For example: the difference between relations and functions; multiple ways of representing functions (e.g., tabular, graphic, symbolic, verbal); properties of functions and relations (e.g., domain, range, continuity); piecewise-defined functions; addition, subtraction, and composition of functions; inverse functions; and graphs of functions and their transformations [e.g., the relationships among $f(x)$, $f(x + k)$, $f(x) + k$, $kf(x)$].

0006 Understand the properties and applications of linear relations and functions.

For example: the relationship between linear models and rate of change; direct variation; graphs of linear equations; slopes and intercepts of lines; finding an equation for a line; algebraic, numeric, and graphical methods of solving systems of linear equations and inequalities; expressions involving absolute value; and using a variety of methods to model and solve problems involving linear functions and systems.

0007 Understand the properties and applications of linear and abstract algebra.

For example: properties of matrices and determinants; representing and solving linear systems using matrices; geometric and algebraic properties of vectors; properties of vector spaces (e.g., basis, dimension); the matrix representing a linear transformation; and the definitions and properties of groups, rings, and fields.

0008 Understand the properties and applications of quadratic relations and functions.

For example: manipulation and simplification of quadratic expressions; methods of solving quadratic equations and inequalities (e.g., factoring, completing the square, quadratic formula, graphing); real and complex roots of quadratic equations; graphs of quadratic functions; relationship between the graphic and symbolic representations of quadratic functions; quadratic maximum and minimum problems; and modeling and solving problems using quadratic relations, functions, and systems.

0009 Understand the properties and applications of polynomial, radical, rational, and absolute value functions and relations.

For example: inverse and joint variation problems; zeros of polynomial functions; manipulating and simplifying polynomial and rational expressions; horizontal and vertical asymptotes; and properties and graphs of and modeling and solving problems involving polynomial, radical, rational, absolute value, and step functions.

0010 Understand the properties and applications of exponential and logarithmic functions and relations.

For example: simplifying exponential and logarithmic expressions; properties and graphs of exponential and logarithmic functions; problems involving exponential growth, decay, and compound interest; applications of logarithmic functions (e.g., decibel scale, Richter scale); and using the inverse relationship between exponential and logarithmic functions to solve problems.

GEOMETRY AND MEASUREMENT [19%]

0011 Understand the principles, concepts, and procedures related to measurement.

For example: unit conversions within and among measurement systems; dimensional analysis; problems involving length, area, volume, mass, capacity, density, time, temperature, angles, and rates of change; degree and radian measure; indirect measurement; the effect of changing linear dimensions on measures of length, area, or volume; and the effects of measurement error and rounding on computed quantities (e.g., area, density, speed).

0012 Understand the axiomatic structure of Euclidean geometry.

For example: the nature of axiomatic systems; undefined terms, postulates, and theorems; relationships among points, lines, rays, angles, and planes; axioms of algebra (e.g., addition postulate), distance and angle measure postulates; special pairs of angles (e.g., supplementary, vertical); properties of parallel and perpendicular lines and planes; triangle congruence conditions; similar triangles; Pythagorean theorem; segments and angles associated with circles; and the origins and development of geometry in different cultures (e.g., Greek, Hindu, Chinese).

0013 Prove theorems within the axiomatic structure of Euclidean geometry.

For example: direct and indirect methods of proof; properties of parallel and perpendicular lines as they relate to polygons and circles; congruent triangles; properties of special triangles; characteristics of parallelograms and other quadrilaterals; similar triangles and other polygons; geometric constructions; and theorems associated with the properties of circles.

0014 Apply Euclidean geometry to solve problems involving two- and three-dimensional objects.

For example: special right triangle relationships; arcs, angles, and segments associated with polygons and circles; properties of three-dimensional figures (e.g., prisms, pyramids, cylinders, cones); perspective drawings and projections; cross sections (including conic sections) and nets; generating three-dimensional figures from two-dimensional shapes; and using two- and three-dimensional models to solve problems.

0015 Understand the principles and properties of coordinate and transformational geometry and characteristics of non-Euclidean geometries.

For example: rectangular and polar coordinates; representation of geometric figures (e.g., lines, triangles, circles) in the coordinate plane; three-dimensional coordinate systems; using concepts of distance, midpoint, slope, and parallel and perpendicular lines to classify and analyze figures (e.g., parallelograms, conic sections); characteristics of dilations, translations, rotations, reflections, and glide-reflections; types of symmetry; transformations in the coordinate plane; and axioms and features of non-Euclidean geometries (e.g., hyperbolic, elliptic).

DATA ANALYSIS, STATISTICS, AND PROBABILITY [10%]

0016 Understand the principles and concepts of descriptive statistics and their application to the problem-solving process.

For example: choosing, constructing, and interpreting appropriate tables, charts, and graphs (e.g., line plots, stem-and-leaf plots, box plots, histograms, circle graphs); measures of central tendency (e.g., mean, median, mode) and dispersion (e.g., range, standard deviation, interquartile range); frequency distributions; percentile scores; and the effects of data transformations on measures of central tendency and variability.

0017 Understand the methods used in collecting and analyzing data.

For example: evaluating real-world situations to determine appropriate sampling techniques and methods for gathering data (e.g., random sampling, avoidance of bias); designing statistical experiments; making appropriate inferences about a population from sample statistics; effects of sample size; interpreting correlation; and problems involving regression models and curve fitting.

0018 Understand the fundamental principles of probability.

For example: probabilities for simple and compound events (e.g., dependent, independent, and mutually exclusive events, conditional probability); the use of simulations to explore probability; probability as a ratio of two areas; and using random variables and probability distributions (e.g., uniform, normal, binomial) to solve problems.

TRIGONOMETRY, CALCULUS, AND DISCRETE MATHEMATICS [16%]

0019 Understand the properties of trigonometric functions and identities.

For example: degree and radian measure; right triangle trigonometry; the laws of sines and cosines; the relationship between the unit circle and trigonometric functions; graphs and properties (e.g., amplitude, period, phase shift) of trigonometric functions and their inverses; trigonometric identities; solving trigonometric equations; and using trigonometric functions to model periodic phenomena.

0020 Understand the concepts of limit, continuity, and rate of change.

For example: limits of algebraic functions and of infinite sequences and series (including the geometric series); continuous and discontinuous functions; the relationship between the secant line and the average rate of change of a function; and solving problems involving average rates of change.

0021 Understand differential calculus.

For example: the slope of the line tangent to a curve; definition and properties of the derivative; differentiability; techniques of differentiation (e.g., product rule, chain rule); the derivative of algebraic and transcendental functions; analyzing the graph of a function; using differentiation to solve problems (e.g., velocity, acceleration, optimization, related rates); verifying that a given function is a solution of a differential equation; and the development of differential calculus.

0022 Understand integral calculus.

For example: algebraic and geometric approximations of the area under a curve; the integral as the limit of a Riemann sum; the fundamental theorem of calculus; techniques of integration; applications of integration (e.g., area, work, volume, arc length, displacement, velocity); and solving differential equations by separation of variables.

0023 Understand the principles of discrete/finite mathematics.

For example: properties of sets; counting techniques (e.g., permutations, combinations); finite differences; the mathematics of finance (e.g., compound interest, annuities, amortization); recursive patterns and relations; iteration; properties of algorithms; linear programming in two variables; properties of matrices; and characteristics and applications of finite graphs and trees.

INTEGRATION OF KNOWLEDGE AND UNDERSTANDING [20%]

In addition to answering multiple-choice items, candidates will prepare written responses to questions addressing content from the preceding objectives, which are summarized in the objective and descriptive statement below.

0024 Prepare an organized, developed analysis emphasizing problem solving, communicating, reasoning, making connections, and/or using representations on topics related to one or more of the following: number sense and operations; patterns, relations, and algebra; geometry and measurement; data analysis, statistics, and probability; trigonometry, calculus, and discrete mathematics.

For example: presenting a detailed solution to a problem involving one or more of the following: place value, number base, and the structure and operations of number systems; properties, attributes, representations, and applications of families of functions; modeling real-world problems with functions; applications of plane and three-dimensional geometry; Euclidean geometry and proof; connections between algebra and geometry; and design, analysis, presentation, and interpretation of statistical surveys.

FORMULAS

Formula	Description
$V = \frac{1}{3}Bh$	Volume of a right cone and a pyramid
$A = 4\pi r^2$	Surface area of a sphere
$V = \frac{4}{3}\pi r^3$	Volume of a sphere
$A = \pi r\sqrt{r^2 + h^2}$	Lateral surface area of a right circular cone
$S_n = \frac{n}{2}[2a + (n - 1)d] = n\left(\frac{a + a_n}{2}\right)$	Sum of an arithmetic series
$S_n = \frac{a(1 - r^n)}{1 - r}$	Sum of a geometric series
$\sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}, \ \lvert r \rvert < 1$	Sum of an infinite geometric series
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Distance formula
$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$	Midpoint formula
$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$	Slope
$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$	Law of sines
$c^2 = a^2 + b^2 - 2ab \cos C$	Law of cosines
$s^2 = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n - 1}$	Variance
$s = r\theta$	Arc length
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Quadratic formula

All examinees taking the Mathematics test (Field 09) will be provided with a Texas Instruments TI 30X Solar Scientific calculator with functions that include the following: addition, subtraction, multiplication, division, square root, percent, sine, cosine, tangent, exponents, and logarithms. **You may NOT bring your own calculator to the test.**

1. **Use the proof below to answer the question that follows.**

Given: $x + z = y + z$

Prove: $x = y$

Proof:

$x + z = y + z$

$(x + z) + (-z) = (y + z) + (-z)$

$x + [z + (-z)] = y + [z + (-z)]$

$x + 0 = y + 0$

$x = y$

Which of the following properties of the real numbers is used to justify one of the steps in the above proof?

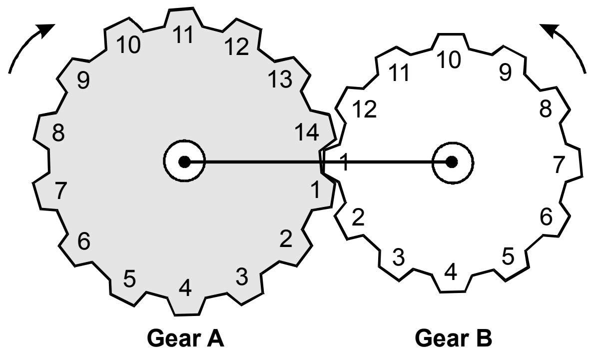
A. associative

B. distributive

C. commutative

D. transitive

2. **Use the diagram below to answer the question that follows.**



Gear A and Gear B rotate as indicated in the diagram above. What is the least number of complete revolutions Gear A must make for the gears to align again as shown?

A. 2

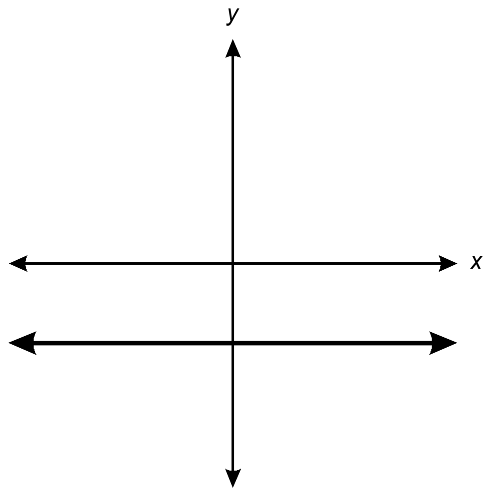
B. 6

C. 42

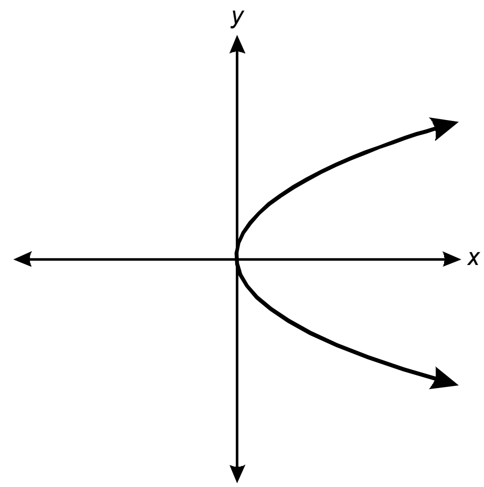
D. 84

3. Which of the following graphs represents a one-to-one function?

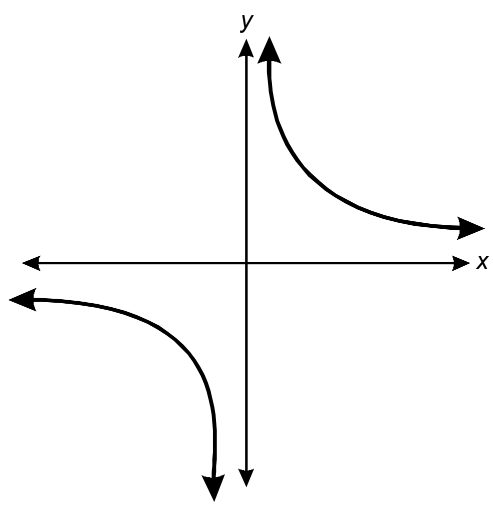
A.



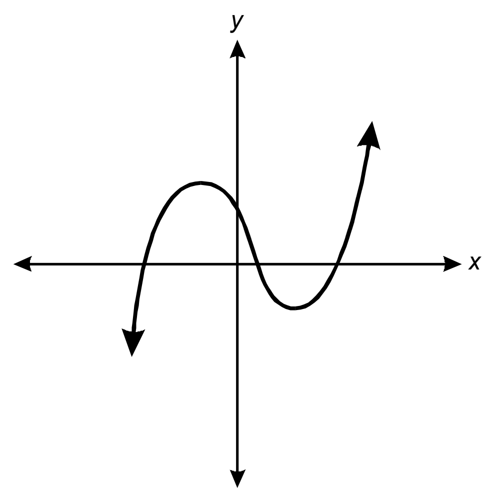
B.



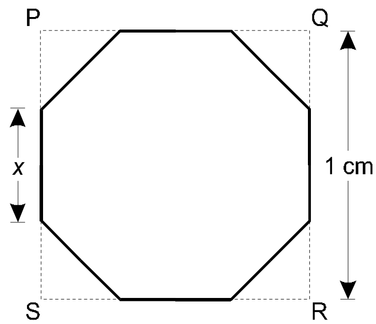
C.



D.



4. **Use the diagram below to answer the question that follows.**



The four corners of a square *PQRS* have been cut off to form a regular octagon, as shown in the diagram above. If each side of the square is 1 cm long, which of the following equations should be solved to find the length of each side of the octagon, x?

A. $x^2 + 2x - 1 = 0$

B. $x^2 - 4x + 2 = 0$

C. $3x^2 + 2x - 1 = 0$

D. $7x^2 - 8x + 2 = 0$

5. If $\log_x 10 = 0.9603$, which of the following is the best approximation of x?

A. 7

B. 9

C. 11

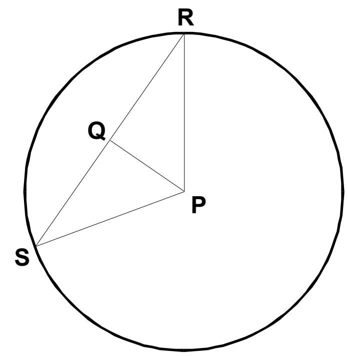
D. 13

6. **Use the geometric proof below to answer the question that follows.**

Given: Circle *P*;

$\overline{PQ}$ is a median of $\triangle PRS$

Prove: $\triangle RPQ \cong \triangle SPQ$



Statements	**Reasons**
1. Circle *P*; $\overline{PQ}$ is a median of $\triangle PRS$	1. Given
2. $\overline{RQ} \cong \overline{QS}$	2. Definition of median
3. $\overline{PR}$ and $\overline{PS}$ are radii of *P*.	3. Definition of radius
4. $\overline{PR} \cong \overline{PS}$	4. All the radii of a circle are congruent.
5. ?	5. ?
6. $\triangle RPQ \cong \triangle SPQ$	6. SSS postulate

Which of the following statements and reasons would be most appropriate in step 5 of this proof?

A. **Statement** $\angle QPR \cong \angle QPS$ — **Reason** $\overline{PQ} \perp \overline{RS}$

B. **Statement** *P* is the center of the circle. — **Reason** Definition of a circle

C. **Statement** $\angle PSQ \cong \angle PRQ$ — **Reason** Properties of isosceles triangle

D. **Statement** $\overline{PQ} \cong \overline{PQ}$ — **Reason** Reflexive property of congruence

7. The center of a town is located at (0, 0) on an x-y coordinate system with each grid unit measured in miles. A radio tower is located at the point (15, –20). The radio signal is strong enough to reach homes within a 40-mile radius. Which of the following inequalities represents all ground locations within 40 miles of the radio tower?

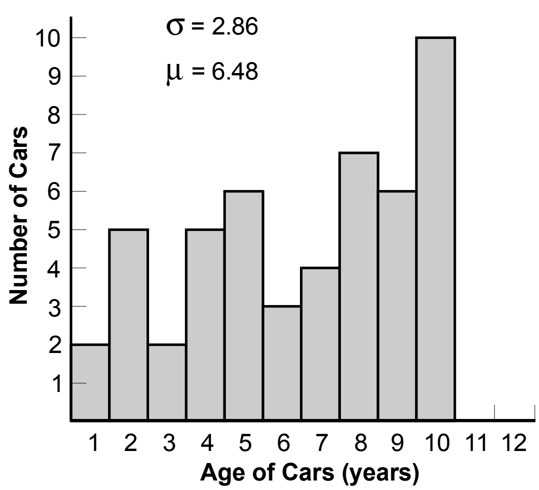
   A. $(x - 15)^2 + (y + 20)^2 \leq 40$

   B. $(x + 15)^2 + (y - 20)^2 \leq 40$

   C. $(x - 15)^2 + (y + 20)^2 \leq 1600$

   D. $(x + 15)^2 + (y - 20)^2 \leq 1600$

8. **Use the diagram below to answer the question that follows.**



   The graph shows the distribution of the age of 50 cars in a parking lot. Rounded to the nearest whole number, the mean, μ, is 6 years and the standard deviation, σ, is 3 years. How many cars have an age within one standard deviation of the mean?

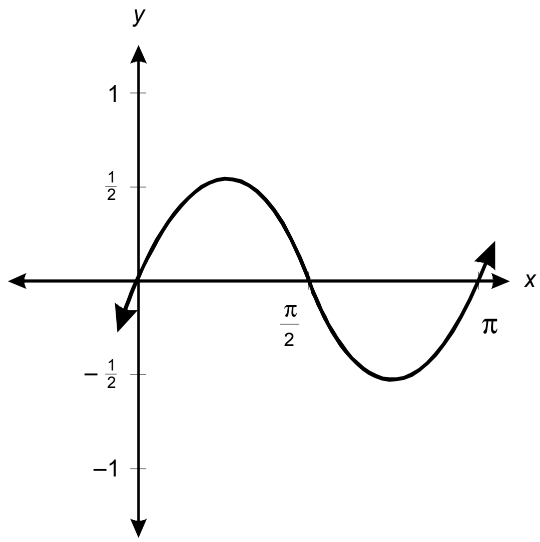
   A. 17

   B. 20

   C. 33

   D. 41

9. **Use the graph below to answer the question that follows.**



The graph of which of the following equations is shown in the figure above?

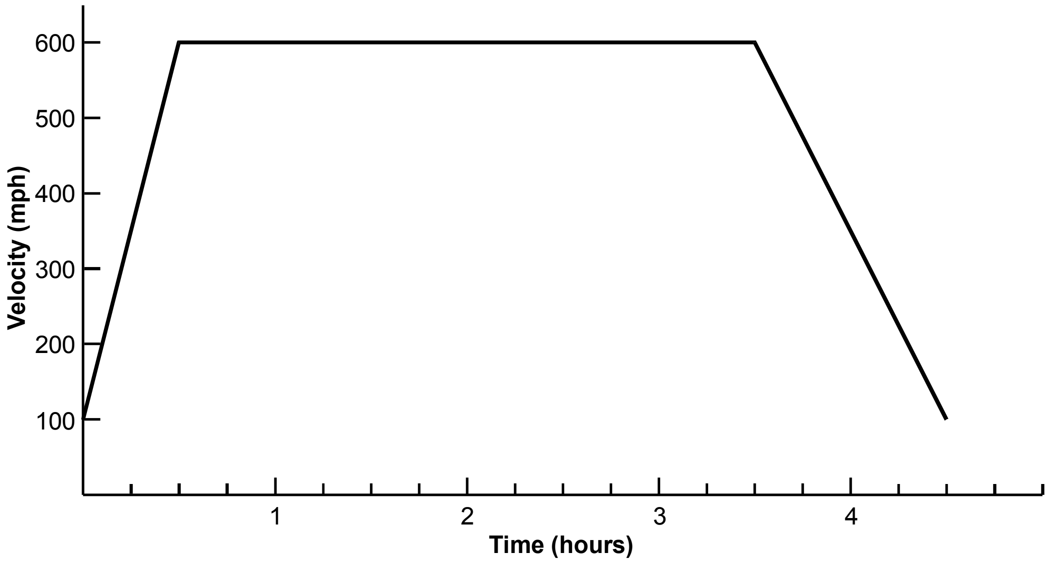
A. $y = \frac{1}{2}\cos(2x)$

B. $y = 2\cos\left(\frac{x}{2}\right)$

C. $y = \frac{1}{2}\sin(2x)$

D. $y = 2\sin\left(\frac{x}{2}\right)$

10. **Use the graph below to answer the question that follows.**



The graph above approximates an airplane's velocity, *v*, in miles per hour, during a four-hour-and-thirty-minute flight. The plane left the ground at a velocity of 100 mph and took 30 minutes to reach its cruising speed of 600 mph. One hour before landing it began to descend and landed at a velocity of 100 mph. Which of the following is the best estimate of the total distance in miles that the airplane flew while in the air?

A. 1875

B. 2175

C. 2250

D. 2325

11. **Use the information below to complete the exercise that follows.**

A tank for watering animals is shaped like a right isosceles trapezoidal prism. The tank is 36 inches across the top and 24 inches deep, has a base of 18 inches, and is 10 feet long. Both the top and the base of the tank are parallel to the ground. The tank initially contains water that is 12 inches deep. Water will be drained from the tank at a rate of 300 cubic inches per minute.

Using your knowledge of two-dimensional and three-dimensional geometry and functions, prepare a response in which you analyze this situation. In your response:

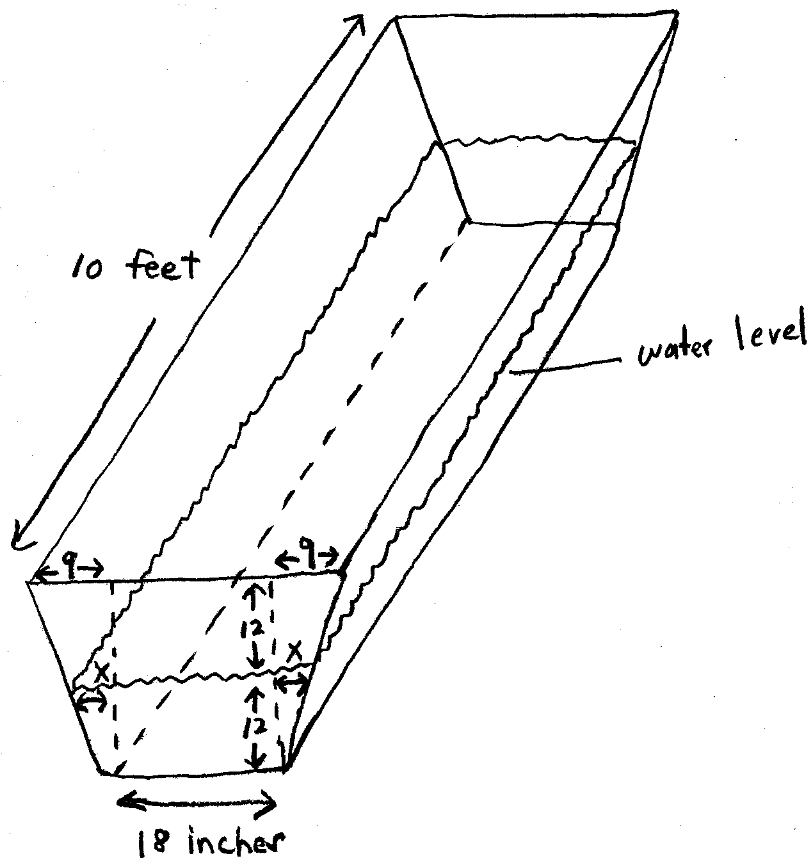
- draw a labeled diagram that clearly presents the given information and any variables needed;
- determine the initial volume of water in the tank;
- derive an equation that expresses the total volume of water in the tank as a function of time;
- sketch a graph of the equation on a set of coordinate axes and describe the meaning of the slope and the y-intercept in the context of the situation; and
- calculate how long it will take to drain the tank.

Be sure to show your work and explain the reasoning you use in analyzing and solving this problem.

Answer Key and Sample Response: Mathematics (09)

Question Number	Correct Response	Test Objective
1.	**A**	Understand the properties of real and complex numbers and the real and complex number systems.
2.	**B**	Understand the principles of number theory.
3.	**C**	Understand the properties of functions and relations.
4.	**A**	Understand the properties and applications of quadratic relations and functions.
5.	**C**	Understand the properties and applications of exponential and logarithmic functions and relations.
6.	**D**	Prove theorems within the axiomatic structure of Euclidean geometry.
7.	**C**	Understand the principles and properties of coordinate and transformational geometry and characteristics of non-Euclidean geometries.
8.	**C**	Understand the principles and concepts of descriptive statistics and their application to the problem-solving process.
9.	**C**	Understand the properties of trigonometric functions and identities.
10.	**D**	Understand integral calculus.

The sample response below reflects a strong knowledge and understanding of the subject matter.



A labeled diagram representing the situation is drawn above.

The water in the tank is in the shape of a prism whose volume can be calculated by multiplying the cross-sectional area of the water in the tank by the length of the tank.

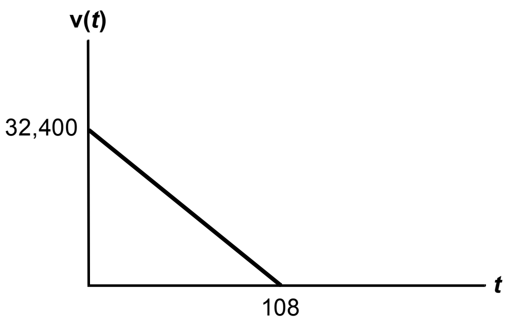
The base of the prism is a trapezoid whose area can be calculated by $A = \frac{1}{2}(b_1 + b_2)h$ with b_1 and b_2 representing the width of the water surface across the tank, and the width of the base of the tank, respectively, and h representing the height of the trapezoid (the depth of the water). In this case, $b_1 = 2x + 18$, $b_2 = 18$, and $h = 12$, so x must be found.

Because of similar triangles $\frac{x}{12} = \frac{9}{24}$, from which x is computed to be 4.5 inches. Thus the width of the water across the tank is 4.5 inches + 18 inches + 4.5 inches = 27 inches. Now find the cross-sectional area of the water in the tank: $A = \frac{1}{2}(27 + 18)12 = 270$ square inches.

The volume of the water is the product of the cross-sectional area of the water in the tank and the length of the tank, or (270 inches)(120 inches) = 32,400 cubic inches. [Note: 10 feet = 120"].

(continued)

Since the water drains at a constant rate of 300 cubic inches per minute, a linear equation can model the situation: $V(t) = 32{,}400 - 300t$ with V representing volume of water in the tank and t representing the time that the tank has been draining.



The above graph represents the function $V(t)$. The slope of the function, −300 cubic inches per minute, represents the rate at which the water is draining from the tank with respect to time. The y-intercept on the graph is 32,400, which represents the quantity of water initially present in the tank.

To find the length of time it will take to drain the tank, set $V(t)$ equal to zero and solve for t. Thus $0 = 32{,}400 - 300t$ implies $-300t = -32{,}400$ and $t = 108$ minutes. It will take 108 minutes to drain the tank.